सत्यं विश्वमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 4th Semester Examination, 2023

MTMHGEC04T/MTMGCOR04T-MATHEMATICS (GE4/DSC4)

ALGEBRA

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions from the following: $2\times5 = 10$

(a) Define a partial order relation. Give an example.

(b) Find the number of elements of order 5 in $\mathbb{Z}_{20}$.

(c) Find all cyclic sub-groups of the group $\{1, i, -1, -i\}$ with respect to multiplication.

(d) Prove or disprove "union of two sub-groups of a group $(G, \circ)$ is a sub-group of $(G, \circ)$".

(e) If G is a group and $a^2 = e$, $\forall a \neq e$. Prove that G is an abelian group.

(f) Is symmetric group S_3 cyclic? Give reasons.

(g) Examine whether $5\mathbb{Z} = \{5x : x \in \mathbb{Z}\}$ is an ideal or not of the ring $(\mathbb{Z}, +, \cdot)$, where $\mathbb{Z}$ is the set of all integers.

(h) Examine if the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & 2b \end{pmatrix} : a, b \in \mathbf{R}\right\}$ contains any divisor of zero.

2. (a) Let a relation R defined on the set $\mathbb{Z}$ by " $a\,R\,b$ if and only if $a-b$ is divisible by 5 for all $a, b \in \mathbb{Z}$. Show that R is an equivalence relation. 4

(b) Show that the set of all permutations on the set $\{1, 2, 3\}$ forms a non abelian group. 4

3. (a) Show that a non-empty subset H of G forms a subgroup of $(G, \circ)$ if and only if (i) $a \in H$, $b \in H \Rightarrow a \circ b \in H$, and $a \in H \Rightarrow a^{-1} \in H$. 4

(b) Prove that $\mathrm{SL}(n, \mathbb{R})$ is a normal subgroup of $\mathrm{GL}(n, \mathbb{R})$. 4

4. (a) Show that every proper sub-group of a group of order 6 is cyclic. 4

(b) Prove that the commutator sub-group of any group is a normal sub-group. 4

5. (a) Prove that the order of every subgroup of a finite group G is a divisor of the order of G. 4

(b) Prove that quotient group of an abelian group is abelian. Is the converse true? Justify. 4

6. (a) Prove that for any positive integer n, the set $U(n)=\{[x] : x$ is positive integer less than n and prime to $n\}$ is a group with respect to 'Multiplication Modulo n'. 4

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}^{+}$ be defined by $f(x)=e^{x}$, $x \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Prove that f is invertible and find f^{-1}. 4

7. (a) Prove that any finite subgroup of the group of non zero complex numbers under multiplication is a cyclic group. 4

(b) Let $G=S_3$ be a group and $H=\{\rho_0, \rho_1, \rho_2\}$ be a subgroup of G. Find all the left cosets of H (where the symbols have their usual meanings). 4

8. (a) Show that the ring of matrices $\left\{\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbf{Z}\right\}$ contains divisors of zeros and does not contain the unity. 4

(b) Let $\mathbb{Z}[\sqrt{2}]=\{a+b\sqrt{2} : a, b \in \mathbb{Z}$ (the set of integers)$\}$. Show that $(\mathbb{Z}[\sqrt{2}], +, \cdot)$ is an integral domain. 4

9. (a) Prove that the ring of matrices $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbf{R}\right\}$ is a field. 4

(b) Prove that a field is an integral domain. 4

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